m	L_m
1	4
2	14
3	194
. . .	. . . . .
10	−115,113,975,804,653,882,052,836,464
20	36,000,517,785,442,762,303,479,300
30	−204,144,540,641,167,292,618,604,303
40	−126,791,709,316,676,382,795,042,761
50	−90,990,560,635,837,660,454,542,648
60	−206,308,592,424,355,282,693,419,690
70	99,498,791,857,820,493,810,407,653
80	269,783,273,665,984,523,074,966,550
. . .	. . . . .
86	−309,403,333,482,440,150,628,882,422
87	−35,184,372,088,832
88	0

Since the first (and only) residue 0 occurs at the 88th term of the above series, it follows, from the foregoing theorem, that $2^{89}-1$, or

618,970,019,642,690,137,449,562,111

is a PRIME NUMBER.

As M. Lucas points out, his method used above is free from any uncertainty as to the accuracy of the conclusion that the number under consideration is prime, in case our attempt to arrive at the residue 0 meets with success, since an error in calculating any term of the series would have the effect of preventing the appearance of the residue 0. We would add that, denoting the number $2^{89}-1$ by N, we have verified that

$3^{N-1}-1$ is divisible by N,

which is in accordance with Fermat's well-known theorem.

DENVER, COLORADO, *June, 1911.*

A GENERALIZATION OF CAUCHY'S FUNCTIONAL EQUATIONS.*

By R. D. CARMICHAEL, Indiana University.

§ 1. INTRODUCTION.

Cauchy† has determined the general continuous solution of each of the simple functional equations

$$f(x+y)=f(x)+f(y),$$
$$f(x+y)=f(x)\,.\,f(y),$$
$$f(xy)=f(x)\,.\,f(y),$$
$$f(xy)=f(x)+f(y),$$

showing that they are, respectively, ax, a^x, x^a, $a\log x$, where in each case a is an arbitrary constant.

Several generalizations of these results have been given. For instance, in the case of the first equation it has been shown‡ that the conclusion remains the same if the hypothesis is weakened by requiring that $f(x)$ shall be continuous only in a finite interval, however small. Or, one may require§ merely that the function shall have an upper limit in an arbitrarily small neighborhood of $x=0$. A generalization of a different kind has been given by H. W. Pexider.‖

In the present paper a generalization of a different type is introduced. Instead of two variables, x and y, of unlimited variation, we retain one variable x of unlimited variation and replace the variable y by a function u_z of the variable z which has the property that, for some set of values of z, u_z may assume at least once each value of an assigned enumerably infinite set of values. Otherwise the function u_z is entirely arbitrary; it need not even be defined at other points. Furthermore, $f(x)$ is required to be continuous only in a given finite interval. The general solution remains the same as before.

This generalization is interesting in that it brings out explicitly the fact that in previous discussions of the problem, as for instance by Cauchy and Darboux, the hypothesis was much stronger than was necessary to enable one to draw the conclusion.

The method employed in the present note (see § 2) is capable of wide use and extension in the study of functional equations of various types. It is due to Monge and Laplace. It consists, abstractly, in reducing the original problem to two steps. The first step consists in finding the general so-

* Read before the American Mathematical Society, October 28, 1911.

† Cauchy, *Œuvres* (2) 3 (1897), p. 98, p. 220.

‡ Darboux, *Mathematische Annalen* 17 (1880), p. 55; Segre, Torino Atti 25 (1890), p. 192. p. 287.

§ La Vallée-Poussin, *Course d'analyse*, 1903, p. 30.

‖ *Monatshefte fur Mathematik und Physik*, 14 (1902), p. 293.

lution of a difference equation which is a special case of the given equation, or is readily deducible from it, this equation having the property that some particular solution of it is the general solution of the original functional equation. The second step consists in determining the arbitrary periodic function or functions of this solution of the difference equation so as to obtain that special solution which satisfies the given functional equation. A like use may also be made of q-difference equations in the solution of functional equations.

In the present paper all quantities are supposed to be real. The extension to the case of complex quantities is not difficult.

§ 2. The Equation $f(x+u_z)=f(x)+f(u_z)$.

We seek to determine completely the function $f(x)$ subject to the conditions that it is to be continuous in the interval 0 to a, where a is any positive or negative constant whatever, and that it is to satisfy the functional equation

$$f(x+u_z)=f(x)+f(u_z), \tag{1}$$

where u_z is a function of the variable z subject to the sole restriction that it is capable of assuming at least once each value in the set

$$2^{-s}n\,a; \quad n, s=0, 1, 2, \ldots, \quad n \leqq 2^s,$$

$u_z=z$ is the simplest such function; and hence equation (1) is a generalization of the Cauchy equation

$$f(x+z)=f(x)+f(z).$$

But the possible functions u_z are evidently of the most varied character. The most general definition of u_z is the following:

Let z_{ns}, $n, s=0, 1, 2, \ldots, n \leqq 2^s$, be any set of numbers no two of which are equal and let

$$u_z=\frac{n\,a}{2^s} \text{ for } z=z_{ns};$$

otherwise let u_z be arbitrary. It need not even be defined for other values of z. Under these conditions we shall show that the solution of (1) is unique except for an arbitrary constant multiplier; in fact, that it is

$$f(x)=ax,$$

where a is an arbitrary constant.

Give to z a value for which u_z is equal to α; then from (1) we have the difference equation

$$f(x+\alpha)=f(x)+f(\alpha). \tag{2}$$

Therefore the general solution of (1) must be some particular solution of (2). But the general solution of (2) is readily found. It is evidently

$$f(x)=ax+p(x) \tag{3}$$

where $a=f(\alpha)/\alpha$ and $p(x)$ is an arbitrary periodic function of period α. Since $f(x)$ is continuous in the interval $(0, \alpha)$, $p(x)$ is continuous in $(0, \alpha)$. Substiiuting this value of $f(x)$ in (1) and reducing, we have

$$p(x+u_z)=p(x)+p(u_z), \tag{4}$$

an equation of the same type as (1) but subject to the further condition that its solution is to be periodic of period α.

Choose z so that $u_z=0$ and take $x=0$. Then from (4) we have

$$p(0)=2p(0);\ \text{or}$$
$$p(0)=0,$$

since $p(x)$ is finite at $x=0$. Hence

$$0=p(0)=p(\alpha)=p(2\alpha)=\ldots=p(n\alpha).$$

Give to z a value such that u_z takes on the value $2^{-s}n\alpha$ and let $x=2^{-s}n\alpha$. Then from (4) we have

$$p\left(\frac{n\alpha}{2^{s-1}}\right)=2p\left(\frac{n\alpha}{2^s}\right).$$

If we employ this as a recursion formula and remember that $p(n\alpha)=0$, we have

$$0=p\left(\frac{n\alpha}{2}\right)=p\left(\frac{n\alpha}{2^s}\right)=\ldots=\left(\frac{n\alpha}{2^s}\right).$$

That is, the function $p(x)$ is equal to zero at every point of the interval

$(0, a)$ which is representable in the form $2^{-s}n\,a$. But the set of points $2^{-s}n\,a$, $n, s=0, 1, 2, \ldots, n \leqq 2^s$ is everywhere dense in the interval $(0, a)$; that is, in every neighborhood (however small) of any point $\bar{a}$ of the interval $(0, a)$ there is a point of the set $2^{-s}n\,a$ different from $\bar{a}$.

Now, if a function is continuous at every point of an interval $(0, a)$ and is equal to zero at every point of a set of points everywhere dense on $(0, a)$, it is clear that the function is equal to zero at every point of the interval $(0, a)$. Hence $p(x)=0$ at every point of the interval $(0, a)$. But $p(x)$ is a periodic function of period a; and therefore $p(x)$, being zero in an interval of length a, is identically zero. Substituting this value of $p(x)$ in (3) we have as the most general solution of (1),

$$f(x)=ax,$$

where a is an arbitrary constant. This result may be stated in the form of the following theorem:

THEOREM I. *Let a be any positive or negative qaantity and let u_z be any function of z capable of assuming at least once each value in the set*

$$2^{-s}n\,a,\ n,\ s=0,\ 1,\ 2,\ \ldots,\ n \leqq 2^s.$$

Then the most general function $f(x)$ which is continuous on the interval $(0, a)$ and satisfies the functional equation

$$f(x+u_z)=f(x)+f(u_z)$$

is

$$f(x)=ax,$$

where a is an arbitrary constant.

§ 3. THE REMAINING CAUCHY FUNCTIONAL EQUATIONS.

The results of the last section we shall now apply to the problem of finding the most general solution of each of the following equations:

$$f(x+u_z)=f(x)f(u_z), \tag{5}$$

$$f(xu_z)=f(x)f(u_z), \tag{6}$$

$$f(xu_z)=f(x)+f(u_z), \tag{7}$$

where the functions are further restricted in each case to satisfy appropriate conditions of continuity on a limited interval. As a special value of u_z we may take $u_z=z$, so that (5), (6), and (7) are generalizations of Cauchy's equations

$$f(x+z)=f(x)f(z),$$
$$f(xz)=f(x)f(z),$$
$$f(xz)=f(x)+f(z).$$

If we take the logarithm of each member of (5) and write $g(v)$ for $\log f(v)$, we have

$$g(x+u_z)=g(x)+g(u_z).$$

Now if $g(x)$ is to be continuous in the interval $(0, a)$ the conditions of theorem I are satisfied for the last equation, so that the general solution is

$$g(x)=ax,$$

where a is an arbitrary constant. Hence

$$\log f(x)=ax.$$

Therefore,
$$f(x)=e^{ax}=\beta^x,$$

where β is an arbitrary constant. If $g(x)$ is continuous in the interval $(0, a)$, $f(x)$ is continuous in the same interval, and conversely, since $f(x)$ cannot be zero on the interval without being identically zero (as the original equation shows). Hence from theorem I we have the following theorem:

THEOREM II. *If a and u_z are defined as in theorem I, then the most general function $f(x)$ which is continuous on the interval $(0, a)$ and satisfies the functional equation*

$$f(x+u_z)=f(x)f(u_z)$$

is
$$f(x)=\beta^x,$$

where β is an arbitrary constant.

In equation (6) let us write

$$x=e^y,\ u_z=e^{v_z},\ f(e^t)=g(t);$$

then we have
$$f(e^{y+v_z})=f(e^y)f(e^{v_z});$$

or
$$g(y+v_z)=g(y)g(v_z).$$

If we assume that v_z satisfies the conditions imposed on u_z in the two previous cases, and that $g(y)$ is continuous in the interval $(0, a)$, we have the hypotheses of theorem II reproduced. Consequently that theorem can be applied to the last equation. If this is done and if the result is stated

in terms of the original function f and the variables x and u_z we have the theorem:

THEOREM III. *Let a be any positive or negative quantity and let u_z be any function of z capable of assuming at least once each value in the set*

$$e^{2^{-s}n\alpha},\ n,\ s=0,\ 1,\ 2,\ \ldots,\ n<2^s.$$

Then the most general function $f(x)$ which is continuous on the interval $(1, e^a)$ and satisfies the functional equation

$$f(xu_z)=f(x)f(u_z)$$

is

$$f(x)=x^r,$$

where r is an arbitrary constant.

For, the most general value of $g(y)$ is

$$g(y)=\beta^y.$$

Hence, $f(e^y)=\beta_y$, since $g(y)=f(e^y)$.

Now put $x=e^y$, whence $y=\log x$; then we have

$$f(x)=\beta^{\log x}=e^{r\log x},$$

where $r=\log\beta$. But $e^{r\log x}=e^{\log x^r}=x^r$. Hence the most general value of $f(x)$ is

$$f(x)=x^r,$$

where r is an arbitrary constant.

If now we take the logarithm of each member of (6) and write $g(v)$ for $\log f(v)$, we have

$$g(xu_z)=g(a)+g(u_z),$$

an equation of the form (7). Hence from theorem III we have at once the following theorem:

THEOREM IV. *If a and u_z are defined as in theorem III, then the most general function of $f(x)$ which is continuous on the interval $(1, e^a)$ and satisfies the functional equation*

$$f(xu_z)=f(x)+f(u_z)$$

is

$$f(x)=r\log x,$$

where r is an arbitrary constant.

THE CYCLIC GROUP AS A BASIC ELEMENT IN THE THEORY OF NUMBERS.

By G. A. MILLER, University of Illinois.

The main objects of the present paper are to point out some of the uses of properties of the cyclic group in studying some of the fundamental theorems of number theory, and to develop these properties of the cyclic group by means of elementary arithmetic considerations, which relate mainly to additive number theory. It is hoped that the paper may serve the double purpose of furnishing an easy but rigorous arithmetic introduction to the cyclic group, and of presenting somewhat novel proofs of wide scope for a few very elementary theorems in number theory.

No knowledge of group theory is presupposed, and the only knowledge of number theory that is presupposed is the fact that a rational integer can be resolved into its prime factors in essentially only one way, together with some familiarity with the elementary notions of a congruence* and of a complete system of residues. Hence the substance and the mode of presentation of this paper would appear suitable for a second chapter of an elementary work on number theory. In fact, it presupposes only a very brief first chapter.

§ 1. Order of a Rational Integer (mod m).

The term *order* of a rational integer a with respect to modulus m, m being a positive rational integer, will be used for the smallest positive rational integer b which is such that the product ab is divisible by m. In other words, the order of a (mod m) is equal to the least number of times that a must be taken as an addend in order to obtain a sum that is divisible by m, and hence the order of a is its period (mod m) as regards addition. It is evident that the order of a (mod m) is equivalent to the product of all the prime factors of m which are not also found in a, and hence *all the rational integers which are congruent to the integer a (mod m) have the same order (mod m)*.

The m successive rational integers,

$$1, 2, 3, \ldots, m, \qquad (A)$$

which represent the complete system of residues (mod m) composed of the smallest possible natural numbers, must therefore represent also numbers of all the possible orders (mod m). All these orders divide m and there is at least one number in (A) whose order is an arbitrary divisor of m.

* Two rational integers a, b are said to be congruent with respect to modulus m whenever $a-b$ is divisible by the rational integer m. This fact is denoted, according to Gauss, by $a \equiv b \pmod{m}$.

Suppose that $m=p_1^{a_1} p_2^{a_2} \ldots p_\lambda^{a_\lambda}$; $p_1, p_2, \ldots, p_\lambda$ being distinct rational prime numbers. A necessary and sufficient condition that a is of order $p_\beta^{a_0}$ (mod m), $1 \leqq \beta \leqq \lambda$, is that a is a multiple of $m \div p_\beta^{a_0}$ which is not divisible by $p_\beta^{a_\beta - a_0 + 1}$. There are, therefore, in (A) exactly $p_\beta^{a_\beta}$ numbers whose orders are powers of p_β, viz., the numbers

$$\frac{m}{p_\beta^{a_\beta}}, 2\frac{m}{p_\beta^{a_\beta}}, 3\frac{m}{p_\beta^{a_\beta}}, \ldots, p_\beta^{a_\beta}\frac{m}{p_\beta^{a_\beta}}.$$

We shall use n_β to represent any one of these $p_\beta^{a_\beta}$ numbers, and hence the sum

$$n_1+n_2+n_3+\ldots+n_\lambda \tag{B}$$

may represent any one of m numbers.

It is not difficult to prove that the totality of the possible numbers represented by (B) is a complete system of residues (mod m). To prove this it is only necessary to show that no two of these numbers are congruent (mod m). In fact, from the congruence

$$n_1+\ldots+n_\beta+\ldots+n_\lambda \equiv n'_1+\ldots+n'_\beta+\ldots+n'_\lambda \pmod{m}$$

it results, if $n_0=n_{\lambda+1}=0$, that

$$n_1+\ldots+(n_\beta - n'_\beta)+\ldots+n_\lambda \equiv n'_1+\ldots+n'_{\beta-1}+n'_{\beta+1}+\ldots+n'_\lambda \pmod{m}.$$

As all these addends, except $n_\beta - n'_\beta$, are divisible by $p_\beta^{a_\beta}$ this number must also divide $n_\beta - n'_\beta$ and hence $n_\beta = n'_\beta$, since all of these numbers may be supposed to be in (A). This proves the following theorem:

If the rational integer m is divisible by exactly λ distinct rational prime numbers, a complete system of residues (mod m) may be obtained by adding successively the numbers of each of the m possible different sets of λ numbers whose orders are powers of the λ different prime factors of m, the numbers having been selected from any complete system of residues (mod m).

To illustrate this theorem we let $m=60=2^2.3.5$. The numbers of (A) whose orders are powers of 2, 3, and 5, respectively, constitute the following rows:

15, 30, 45, 60
20, 40, 60
12, 24, 36, 48, 60

It is not difficult to verify that the 60 different sums obtained by taking one and only one addend from each of these three rows constitutes a complete system of residues (mod 60).

Suppose that p_β^α is the highest power of p which divides the order of a (mod m), and that the order of b (mod m) is not divisible by p_β^α. It results directly from the definition of order that the highest power of p_β which divides a is ${}_\beta^\alpha\beta^{-\alpha}$ and that b is divisible by a higher power of p_β. Hence $a+b$ is divisible by $p_\beta^\alpha{}^{-\alpha}$ but not by the $(\alpha_\beta - \alpha + 1)$th power of p_β, since the latter of these numbers divides b but does not divide a. This proves the theorem:

If p^α is the highest power of a rational prime number which divides the order (mod m) of one of two rational integers without dividing this order of the other, then the order (mod m) of the sum of these integers is divisible by p^α but not by $p^{\alpha+1}$.

From this theorem we can readily deduce the following useful corollary: *If the orders of two or more rational integers are relatively prime, the order of their sum is the product of their orders, all orders being taken with respect to the same modulus.*

The number of the numbers in (A) whose orders are m is denoted by $\phi(m)$ according to Gauss,* and this number is called the *Euler function* of m because Euler first found a general formula for its value. This number has also been called the *indicator* of m by Cauchy, and the *totient* of m by Sylvester. These three terms for the same function of m are still in common use. It is evident that $\phi(m)$ is the total number of the different sets, each set being composed of all congruent numbers (mod m), which involve the numbers of order m (mod m). When $m=p^\alpha$, p being a prime number, it is easy to see that

$$\phi(\mathrm{m})=\phi(p^\alpha)=p^\alpha - p^{\alpha-1}=p^{\alpha-1}(p-1)=p^\alpha(1-1/p),$$

since only $p^{\alpha-1}$ of the numbers of (A) are divisible by p when $m=p^\alpha$, viz., the numbers,

$$p,\ 2p,\ 3p,\ \ldots,\ p^{\alpha-1}.p.$$

From this formula it results that exactly $p_\beta^{\alpha\beta}(1-1/p_\beta)$ of the given n_β numbers in (A) are of order $p_\beta^{\alpha\beta}$, $1 \leqq \beta \leqq \lambda$. The complete system of residues (mod m)

$$n_1+n_2+n_3+\ldots+n_\lambda \qquad \text{(B)}$$

must therefore contain exactly

Disquisitiones Arithmeticae, 1801, Art. 38; German edition, 1889, p. 22.

$$p_1^{a_1}(1-\frac{1}{p_1}).p_2^{a_2}(1-\frac{1}{p_2}).p_3^{a_3}(1-\frac{1}{p_3})\ldots p_\lambda^{a_\lambda}(1-\frac{1}{p_\lambda})$$

$$=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_\lambda})$$

numbers of order m, since the order of the sum of such a set of λ numbers is equal to the product of their orders according to the given theorem, and the choice of one of these λ numbers does not restrict the choice of another. We have therefore established Euler's formula,

$$\phi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_\lambda}).$$

The number of numbers of order p_β^γ, $0 \leqq \gamma \leqq a_\beta$, in n_β is $\phi(p_\beta^\gamma)$ since n_β contains exactly p_β^γ numbers whose orders divide p_β^γ and $1/p_\beta$ of these have orders which divide $p_\beta^{\gamma-1}$. Since all the numbers of (A) whose orders divide d are multiples of m/d it results that (A) contains exactly d numbers whose orders divide d if d is an arbitrary divisor of m. These d numbers include exactly $\phi(d)$ numbers of order d in accord with what was proved above. As all the numbers of (B) have an order which divides m it results from this that

$$m=\phi(d_1)+\phi(d_2)+\ldots+\phi(d_l),$$

where $d_1, d_2, \ldots, d_l$ are all the positive integral divisors of m, including 1, and m, and hence the second member of this equation gives the sum of the numbers of the integers of the same order in a complete system of residues (mod m).

If $m=m_1m_2$, where m_1, m_2 are relatively prime, we may assume that the orders of numbers in $n_1, n_2, \ldots, n_\delta$ divide m_1, while those in $n_{\delta+1}, \ldots, n_\lambda$ divide m_2. Hence there are $\phi(m_1)$ numbers of order m_1 in $n_1+n_2+\ldots+n_\delta$ and there are $\phi(m_2)$ numbers of order m_2 in $n_{\delta+1}+\ldots+n_\lambda$. The sum of any one of the former $\phi(m_1)$ numbers and any one of the latter $\phi(m_2)$ numbers is of order m since these orders are relatively prime. Since this sum can be formed in $\phi(m_1).\phi(m_2)$ ways and the number of ways in which this sum can be formed is also $\phi(m)$, it results that $\phi(m)=\phi(m_1)\phi(m_2)$. This formula could also have been deduced directly from Euler's formula,

$$\phi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_\lambda}).$$

§ 2. The Cyclic Group of Order m.

If an operation s has the period m it is said to generate the *cyclic group* of order m, and it may be represented by the different powers of s, as follows:

$$s, s^2, s^3, \ldots, s^m.$$

The exponents of s form the complete set of residues (mod m) which was denoted by (A) in the preceding section. As the combination* of two or more of these operations is effected by the addition of their exponents (mod m), it results that the abstract properties of the cyclic group are exhibited by the laws governing the addition (mod m) of the numbers in the set

$$1, 2, \ldots, m.$$

From this it follows directly that there is one and only one cyclic group of order m. From the fact that (A) contains exactly $p_\beta^{\alpha_\beta}$ numbers whose orders are powers of p_β, it results that the cyclic group of order m contains exactly $p_\beta^{\alpha_\beta}$ operators whose orders divide $p_\beta^{\alpha_\beta}$. The fact that the numbers

$$n_1+n_2+\ldots+n_\lambda \tag{B}$$

form a complete system of residues is equivalent to the fact that every element or operator or operation of a cyclic group is the product, in one and only one way, of elements whose orders are powers of prime numbers, no two of these orders being powers of the same prime.

The fact that there are exactly d numbers in (A) whose orders divide d, if d is an arbitrary divisor of m, is equivalent to the fact that a cyclic group contains one and only one subgroup whose order is an arbitrary divisor of the order of the group; and the formula

$$m=\phi(d_1)+\phi(d_2)+\ldots+\phi(d_l),$$

where $d_1, d_2, \ldots, d_l$ are all the different divisors of m, including 1 and m, merely asserts that if the numbers of all elements of the various different orders in a cyclic group are added together their sum is the order of this cyclic group.

The fact that $\phi(m)=\phi(m_1)\phi(m_2)$, if $m=m_1m_2$ and m_1, m_2 are relatively prime, is included in the theorem that a commutative group whose order is divisible by more than one prime is the product of its subgroups of

* This combination is commonly called finding the product of the operations involved.

relatively prime orders, and the order of the product of two commutative elements of a group is equal to the product of the order of these elements whenever these orders are relatively prime. The latter theorem is evidently a special case of the theorem, if the order of one of two commutative elements of a group is divisible by a higher power of a given prime than the order of the other then the order of the product of these two elements is also divisible by this higher power of the given prime, and the highest power of this prime which divides the order of the former element is also the highest power of this prime which divides the order of this product.

These parallel theorems may suffice to exhibit the fact that some of the most fundamental theorems in number theory are also fundamental theorems in group theory, and it seems unfortunate that our elementary books on number theory do not exhibit these points of contact more fully. It should be emphasized that the developments of § 1 are not as simple as they would have been if the properties of the cyclic group had been first developed in the well known manner and if these had been employed in the proof of the given theorems.

The present mode of procedure has been adopted because of the fact that the cyclic group is so fundamental that it seems desirable to establish its fundamental properties in more than one way, and these propertios will doubtless appear more significant if they have been reached by different routes. It may also serve to illustrate relations between additive and multiplicative number theory. In fact, group theory owes a considerable part of its usefulness to the fact that it establishes close contact between additive and multiplicative number theory, as is fully illustrated by the preceding developments.

It may be added that the formula for the $\phi_r(m)$, viz.,

$$\phi_r(m)=m^r(1-\frac{1}{p_1{}^r})(1-\frac{1}{p_2{}^r})\ldots(1-\frac{1}{p^r})$$

results directly from the fact that this formula gives the number of operators of order m in the direct product of r cyclic groups of order m, and hence also in the direct product of r, λ cyclic groups of which there are r of each of the orders, $p_1{}^{a_1}, p_2{}^{a_2}, \ldots, p_\lambda^{a_\lambda}$. The number of operators of order $p_1{}^{a_1}$ in the direct product of the first r of these subgroups is clearly

$$p_1{}^{a_1 r}-p_1{}^{a_1 r-r}=p_1{}^{a_1 r}(1-1/p_1{}^r),$$

and from this fact the given formula results immediately.*

* Cf. The American Mathematical Monthly, Vol. 11, (1904), p. 129.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

CALCULUS.

368. Proposed by C. N. SCHMALL, New York City.

Prove, by calculus, that of all isoperimetric triangles, the equilateral has the greatest area.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Letting s represent the half sum of the three sides x, y, z, we have for the square of the area, $s(s-x)(s-y)(s-z)$; therefore $m=(s-x)(s-y)(s-z)=(s-x)(s-y)(x+y-s)=xy(x+y)-(x^2+3xy+y^2)s+2(x+y)s^2-s^3$ is to be a maximum. By differentiation

$$\frac{\partial m}{\partial x}=y(2x+y)-(2x+3y)s+2s^2,$$

$$\frac{\partial m}{\partial y}=x(2y+x)-(2y+3x)s+2s^2,$$

From $\frac{\partial m}{\partial x}=0$, $\frac{\partial m}{\partial y}=0$,

by subtracting one from the other, we get $(x-y)[s-(x+y)]=0$; whence $x-y=0$, the other root $x+y=s$ not being available. Hence $y=0$.

Substituting in either one, we have $3x^2-5xs+2s^2=0$; whence $x=s$, $x=\frac{2}{3}s$, the former of which is unavailable.

$\therefore x=\frac{2}{3}s$, $y=\frac{2}{3}s$, and $z=\frac{2}{3}s$.

II. Solution by the PROPOSER.

Let the given perimeter be $2s$; let the sides be x, y, z, and u the area. Then we have $z=2s-x-y$. Now

$$u=\sqrt{[s(s-x)(s-y)(x+y-s)]}\ldots(1),$$

which is to be rendered a maximum. Taking logarithms, we have

$$2\log u=\log s+\log(s-x)+\log(s-y)+\log(x+y-s).$$

$$\therefore \frac{2}{u}\frac{\partial u}{\partial x} = -\frac{1}{s-x} + \frac{1}{x+y-s}. \qquad \therefore \frac{\partial u}{\partial x} = \frac{u}{2}\frac{2s-2x-y}{(s-x)(x+y-s)} \ldots (2),$$

$$\text{and } \frac{\partial u}{\partial y} = \frac{u}{2}\frac{2s-2y-x}{(s-y)(x+y-s)} \ldots (3).$$

Equating (2) and (3) to 0, we have $2s-2x-y=0$, $2s-2y-x=0$, whence $x=\frac{2}{3}s$, $y=\frac{2}{3}s$, $z=2s-x-y=\frac{2}{3}s$.

Hence the triangle is equilateral; and there is evidently a maximum.

309. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology.

In practical problems involving maxima and minima, it is really the greatest and least values of the function which are desired. Show why we can assume that the maximum is the greatest value and the minimum the least value under the conditions.

No solution of this problem has been received.

310. Proposed by C. N. SCHMALL, New York City.

Evaluate $\int_0^\pi \frac{dx}{1-2a\cos x+a^2}$. Edwards' *Integral Calculus for Beginners*, page 131, ex. 9, (iii). The answer given is $\frac{\pi}{1-a^2}$. Is this a complete answer to the question?

I. Solution by H. PRIME, Boston, Mass.

$$\int_0^\pi \frac{dx}{1+a^2-2a\cos x} = \left[\frac{2}{\pm(1-a^2)}\tan^{-1}\left(\pm\frac{1+a}{1-a}\tan\frac{x}{2}\right)\right]_0^\pi$$

$$= \frac{(2n+1)\pi}{\pm(1-a^2)} - \frac{2n\pi}{\pm(1-a^2)} = \frac{\pi}{\pm(1-a^2)}.$$

It should be noted that if $a=\pm 1$, the function $1/(1+a^2-2a\cos x)$ becomes infinite for $x=0$ or $x=\pi$. Hence, in this case, the integral has no meaning for the required limits, as they do not exist. The result also shows this for $\pi \div (1-a^2)=\infty$, when $a=\pm 1$.

Also solved by J. Scheffer, Francis Rust, A. M. Harding, and V. M. Spunar.

311. Proposed by WILMER THOMPSON, Senior, Drury College.

Solve the differential equation,

$$\left(\frac{dy}{dx}\right)^3 + x^3 = ax\left(\frac{dy}{dx}\right).$$

[From Forsythe's *Differential Equations*, p. 47.]

I. Solution by W. W. BEMAN, Professor of Mathematics, University of Michigan, and A. H. HOLMES, Brunswick, Maine.

$p^3+x^3=axp$. Putting $p=xu$, $x=\frac{ay}{1+u^3}$, $p=\frac{au^2}{1+u^3}$.

Further, $p=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{(1+u^3)^2}{a(1-2u^3)}\cdot\frac{dy}{du}$.

$\therefore \frac{dy}{du}=a^2\frac{(1-2u^3)u^2}{(1+u^3)^3}$. Integrating, $y+c=\frac{a^2}{6}\cdot\frac{1+4u^3}{(1+u^3)^2}$.

Eliminating u from this equation and $x=\frac{au}{1+u^3}$,

$$216(y+c)^3-36a^2(y+c)^2-108ax^3(y+c)+16a^3x^3+27x^6=0.$$

II. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Expressing the above equation in the form, $p^3-axp+x^3=0$, and solving for p by Cardan's Method, we obtain

$$p=\left(-\frac{x^2}{2}+\sqrt{+\frac{x^6}{4}-\frac{a^3x^3}{27}}\right)^{\frac{1}{3}}+\left(-\frac{x^3}{2}-\sqrt{+\frac{x^6}{4}-\frac{a^3x^3}{27}}\right)^{\frac{1}{3}}\ldots(1).$$

Assuming $-\frac{a^3x^3}{27}=-\frac{x^6}{4}\sin^2\theta\ldots(2)$, (1) becomes

$$p=(-x)[(\cos\tfrac{1}{2}\theta)^{\frac{2}{3}}+(\sin\theta)^{\frac{2}{3}}]\ldots(3), \text{ and } x=\frac{4^{\frac{1}{3}}a}{3(\sin\theta)^{\frac{2}{3}}}\ldots(4).$$

From (4), $dx=-\frac{2.4^{\frac{1}{3}}a}{9}\frac{\cos\theta}{(\sin\theta)^{\frac{1}{3}}}d\theta$. By substituting in (1),

$$p=\frac{2.16^{\frac{1}{3}}a^2}{27}\left[\frac{(\cos\frac{1}{2}\theta)^{\frac{2}{3}}}{(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta)^{\frac{2}{3}}}+\frac{(\sin\frac{1}{2}\theta)^{\frac{2}{3}}}{(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta)^{\frac{2}{3}}}\right]\frac{(2\cos^2\frac{1}{2}\theta-1)}{(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta)^{\frac{1}{3}}}$$

$$dy=\frac{2.16^{\frac{1}{3}}.a^2}{27.2^{\frac{1}{3}}}\left[\frac{2(\cos\frac{1}{2}\theta)^{\frac{1}{3}}}{(\sin\frac{1}{2}\theta)^{\frac{1}{3}}}-\frac{1}{(\sin\frac{1}{2}\theta)^{\frac{1}{3}}(\cos\frac{1}{2}\theta)^{\frac{1}{3}}}+\frac{2}{(\sin\frac{1}{2}\theta)^{\frac{1}{3}}(\cos\frac{1}{2}\theta)^{\frac{1}{3}}}\right.$$

$$\left.-\frac{1}{(\sin\frac{1}{2}\theta)^{\frac{1}{3}}(\cos\frac{1}{2}\theta)^{\frac{1}{3}}}\right]d\theta.$$

Placing $\tan\theta=t$, $d\theta=\frac{dt}{1+t^2}$, $\sin\theta=\frac{t}{(1+t^2)^{\frac{1}{2}}}$, and $\cos\theta=\frac{1}{(1+t^2)^{\frac{1}{2}}}$.

Integrating, we obtain,

$$y+c=-\tfrac{3}{4}[(\tan\tfrac{1}{2}\theta)^{-4}+(\tan\tfrac{1}{2}\theta)^{4}]\frac{2.16^{\frac{1}{3}}a^3}{27.2^3}=-\frac{a^2}{36}\left[\left(\tan\tfrac{1}{2}\theta\right)^{-4}+\left(\tan\tfrac{1}{2}\theta\right)^{4}\right]$$

from which, by substitution, the solution may be completed.

III. Solution by WILLIAM HOOVER, Athens, Ohio.

Let $dy/dx=p$, as usual; then $p^3-axp+x^3=0$.

The condition for equal values of p from this cubic is

$$\frac{p^6}{4}=\frac{a^3\,p^3}{27},\text{ or }p=\frac{dy}{dx}=\frac{a}{3}\sqrt[3]{4}.$$

Integrating, $\frac{3y}{a\sqrt[3]{4}}=x+c$, or, $27y^3=4a^3(x+c)^3$.

Also solved by S. G. Barton.

MECHANICS.

255. Proposed by the late G. B. M. ZERR, Ph. D.

Assuming the resilience of volume of mercury to be constant at all depths and to be 54.20×10^{10} in *C. G. S.* units and that a mile = 160933 centimeters. Find the depth of an ocean of mercury at a point where its density is double the surface density, 13.596.

Solution by B. F. FINKEL, Ph. D., Drury College.

Let x_0 be the length of a column of mercury of one square centimeter cross-section and uncompressed, such that its weight is sufficient to compress a cubic centimeter to half its volume, which is equivalent to doubling its density.

By Hooke's law, $p=E\frac{\triangle v}{v}=\frac{1}{2}E$, where E is the coefficient of elasticity. But $p=13.596gx_0$.

$\therefore x_0=\frac{E}{27.192g}$. Let $\triangle x_0$=the amount of compression of the column by virtue of its own weight, and let x=the length of any portion of the column measured from the surface downward, and let dx be an element of length of this column. Then the strain, s, in this element is

$s = \frac{d(\triangle x_0)}{dx}$. But by Hooke's Law, $s = \frac{p}{E}$. Hence, $\frac{d(\triangle x_0)}{dx} = \frac{p}{E}$

$= \frac{13.596gx}{E}$, or $d(\triangle x_0) = \frac{13.596gxdx}{E}$, and

$$\therefore \triangle x_0 = \frac{13.596g}{E}\int_0^{x_0} x dx = \frac{13.596g(\frac{1}{2}x_0^2)}{E} = \frac{6.798gx_0^2}{E}.$$

$\therefore$ The depth of the ocean required is $x = x_0 - \triangle x_0$

$$= \frac{E}{27.192g} - \frac{6.798g}{E}\left(\frac{E}{27.192g}\right)^2 = \frac{E}{27.192g}\left[1 - \tfrac{1}{4}\right]$$

$$= \frac{3E}{108.768g}\text{cm.} = \frac{3E}{108.768 \times 160933 \times g}\text{mi.}$$

Using $g = 981$, and $E = 33 \times 10^{10}$, we have $x = 61.8$ miles.

REMARK. We are unable to get data from which to obtain Dr. Zerr's value for the volume resilience. The value of E as given in Kimball's *Physics*, page 158, and which we have used in the above solution, is very different from what would be obtained from Dr. Zerr's value for volume resilience.

NOTES AND NEWS.

The fifth regular meeting of the Southwestern Section of the American Mathematical Society was held at Washington University, St. Louis, Missouri, on Saturday, December 2, 1911. S.

The eighteenth summer meeting of the American Mathematical Society was held at Poughkeepsie, New York, on September 12-13, 1911. There were about thirty-five members in attendance and twenty-six papers were presented. S.

The twenty-ninth regular meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, December 29, 30, 1911. Titles and abstracts of papers to be presented at this meeting should be sent to the secretary of the Section, Professor H. E. Slaught, the University of Chicago, Chicago, Illinois. S.

Dr. Jacques Hadamard, Professor of Analytic and Celestial Mechanics in the Collége de France, lectured at Columbia University five times each week in October. He also delivered two lectures at the University of Chicago on October 31 on the topics: (1) Certain Mathematical Improvements likely to be useful in the study of Physics; (2) Psychology of Mathematicians. S.

We learn from *Science* that the Jean Reynaud prize of ten thousand francs, awarded by the Paris Academy of Sciences every five years, has been bestowed this year on Professor Emile Picard, for his contributions to mathematics. And from the same journal we learn that the DeMorgan medal of the London Mathematical Society has been awarded to Professor Horace Lamb, F. R. S., for his research in Mathematical Physics. F.

Editor Miller, in *Science* for December 1, has pointed out a number of mathematical errors in the eleventh edition of *Encyclopedia Britannica.* Among the errors pointed out by Professor Miller are, (1) the incorrect statement of the origin of *zero*, (2) the first use of the term *abscissa*, and (3) the first use of $\phi(n)$. We remark in passing that in our judgment the eleventh edition of this monumental work is not up to the standard of excellence for the mathematician as is the ninth edition. F.

The annual conference of the University of Chicago with the secondary schools, which has heretofore been held in November, has been postponed till March, 1912, in order that a comprehensive plan of visitations may be carried on during the autumn and winter, both by secondary teachers at the University and by instructors in the University among the schools. The reports of these visitations are to form the basis of the next conference and it is expected that much mutual benefit will be derived from such a scheme of co-operation between the schools and the University. S.

Several of the reports of the American committees of the International Commission on the Teaching of Mathematics have been published by the United States Bureau of Education, and are not being distributed by the bureau. These reports are of the greatest value to teachers of mathematics and should be eagerly sought by them.

There has come to us the report of committees Nos. V, VII, IX, X, XII. Committee No. V made a report on the Training of Teachers of Elementary and Secondary Mathematics; Committee No. VII, on Examinations in Mathematics Other Than Those Set by the Teacher for His Own Classes; Committee No. IX, Mathematics in the Technological Schools of Collegiate Grade in the United States; committee No. X, Undergraduate Work in Mathematics in Colleges of Liberal Arts and Universities; and Committee No. XII, Graduate Work in Mathematics in Universities and in Other Institutions of Like Grade in the United States. F.

The one hundred and fifty-fifth regular meeting of the American Mathematical Society was held at Columbia University, New York City, on Saturday, October 28, 1911. S.

The preliminary report of the National Committee of fifteen on Geometry Syllabus was the subject of discussion at the autumn meeting of the Middle States and Maryland Association of teachers of mathematics. It was also discussed at the meeting of the mathematics section of the high school teachers of Chicago, November 11, 1911. There is a constant demand for copies of this report which cannot be supplied until the committee complete the revision at which they are now working, when another edition will be printed for general distribution and a regular sale price will be determined. S.

One of the editors desires to put on record a dream in regard to mathematical notation. He thought he was visiting a class in elementary mathematics and heard the instructor say "x sub sucker r from 0 to 10." Several students inquired what he meant and he explained that this was his way of expressing the relation x_r, $0 \leqq r \leqq 10$, as r was between the suckers $<$, $<$. Moreover, this appeared to be in accord with the statement in the Scriptures, "Whosoever hath, to him shall be given, and he shall have more abundance; but whosoever hath not, from him shall be taken away even that he hath." The symbol $<$ seemed to suck from the smaller towards the larger. M.

The last annual meeting of the Central Association of Science and Mathematic Teachers was held at Lewis Institute, Chicago, Illinois, on November 30 and December 1, 1911. This association has for years been carrying forward progressive and consistent work along all lines of science and mathematics. In the latter it has developed reports on algebra, geometry, and unified mathematics, which have had widespread influence all over the country, but especially in the middle West. Every secondary teacher of any of these subjects within this territory should be allied with this association. *School Science and Mathematics* is the official publication of this association. S.

The annual conference of the University of Illinois with the secondary schools was held on Thursday, Friday, Saturday, November 23-25, 1911. An important report of a committee on Geometry was made at this meeting. The University of Illinois has conducted for several years, in co-operation with the high schools, a careful study of the secondary curriculum, with a view to formulating programs of study in the various subjects. Two years ago an excellent report on Algebra was adopted and incorporated in the official publication of the conference. Last year the Geometry report in preliminary form was the topic in two long sessions of the Mathematical Section. This year it was presented in final form. S.

BOOKS.

The Teaching of High School Mathematics, by George W. Evans, Headmaster of the Charlestown High School, one of the Riverside Educational Monographs, edited by Henry Suzzallo, Professor of the Philosophy of Education, Teachers College, Columbia University: 8vo. Cloth, viii+94 pages. Price, 35 cents. Boston: Houghton, Mifflin & Co.

In addition to the Editor's Introduction, the book contains a brief discussion of the following: The Modern Point of View; The Order of Topics; Equations and their Use; Some Rules of Thumb; Geometry as Algebraic Material; The Graphical Method; The Bases of Proof in Geometry; The Method of Limits; Simpson's Rule and the Curve of Sections; The Teacher. The Author presents some very helpful and suggestive thoughts on elementary teaching and his book is worthy a careful study by the teacher of elementary mathematics. F.

A College Text-Book of Physics. By Arthur L. Kimball, Ph. D., Professor of Physics in Amherst College. 8vo. Cloth, ix+692 pages. Price, $2.75. New York: Henry Holt & Co.

This is one ofthe most illuminating texts on physics for college use that has appeared in recent years. The method of presentation and the clearness of exposition is very attractive and will appeal strongly to those teachers who want a good text with not too much mathematical development of physical principles. F.

Plain and Solid Geometry with Problems and Applications. By H. E. Slaught, Ph. D., Associate Professor of Mathematics in the University of Chicago and N. J. Lennes, Ph. D., Instructor in Mathematics in Columbia University. 8vo. Cloth, xii+470 pages. Boston: Allyn & Bacon.

In writing this book, the authors were guided by the following two purposes, viz: (a) that pupils may gain by gradual processes the power and habit of deductive reasoning and (b) that they may learn to know the facts of elementary geometry as elementary properties of the space in which they live.

They have made use freely of the best of the many suggestions offered by various committees appointed at various times during the past five or six years, for the improvement of the teaching of geometry. Many problems of very practical applications have been incorporated in the work, and it is believed that these will stimulate an interest in the minds of most students who use the book. The book has received the most hearty indorsement by the educational public, a fact evinced by the many schools in which it was adopted immediately after coming from the press. F.

Vector Analysis. An introduction to Vector-Methods and their various applications to Physics and Mathematics. By Joseph George Coffin, B. S., Ph. D., (Massachusetts Institute of Technology, '98, and Clark University, '03) Instructor in Physics at the College at the City of New York. Second edition. 12 mo. Cloth, xviii+262 pages. Price, $2.50. New York: John Wiley & Sons.

The cordial reception of the first edition of this book both by American and European Mathematicians has encouraged the author to bring out this second edition. Certain portions of the work have been rewritten and 14 pages of notes have been added to the appendix. The book is doing great good in leading teachers of physics to an understanding of the use of a very powerful instrument for physical research. F.

An Introduction to Mathematics,. By A. N. Whitehead, S. C. D., F. R. S. Author of "Universal Algebra." 12mo. Cloth, 256 pages. Price, 75 cents. New York: Henry Holt & Co.

"Mr. Whitehead sets out not to teach mathematics, but to enable students from the very beginning of their course to know what the science is about and why it is necessarily the foundation of exact thought as applied to natural phenomena? It is just because mathematical ideas are abstract that they supply what is wanted for scientific description of the course of events, freed from reference to particular persons or particular types of sensation. From this starting-point the author proceeds to explain the true inwardness of 'variables,' 'dynamics,' the symbolism of mathematics, generalizations of numbers, imaginary numbers, co-ordinate geometry, conic sections, functions, periodicity, trigonometry, the differential calculus, and geometry. An admirably clear exposition, illustrated throughout with diagrams."

Analytical Mechanics, Comprising the Kinetics and Statics of Solids and Fluids. By Edwin H. Barton, D. Sc. (Lond.), F. R. S. E., F. Ph. S. L., Professor of Experimental Physics, University College, Nottingham. 8vo. Cloth, xx+535 pages, with diagrams. London: Longmans, Green & Co.

This book, which requires on the part of the student an elementary knowledge of the calculus, gives a fairly complete treatment of the kinetics and statics of solids and of fluids, closing with a brief chapter on elasticity.

The author, after a brief introduction, takes up the treatment of mechanics under the following: kinematics, kinetics, statics, hydromechanics, and elasticity, and throughout the body of the work are scattered at frequent intervals sets of examples; while at the end is a list of miscellaneous problems of various degrees of difficulty.

The treatment of the various subjects is very clear and direct and well adapted to the needs of the students of engineering. F.

Physics for College Students. By Henry S. Carhart, LL. D., Professor Emeritus of Physics, University of Michigan. 8vo. Cloth, viii+622 pages. Boston: Allyn & Bacon.

The author informs us that this book was prepared in response to many inquiries for a text somewhat more advanced than his High School Physics and distinctively less mathematical than his *University Physics*. We believe that Professor Carhart has succeeded in preparing a book at once scientific and teachable and that the treatment here presented will lead to satisfactory results. F.

College Physics. By John Oren Reed, Ph. D., Professor of Physics in the University of Michigan and Dean of the Department of Literature, Science, and the Arts and Karl Eugene Guthe, Ph. D., Professor of Physics in the University of Michigan. 8vo. Coth, xxviii+622 pages. New York: Macmillan & Co.

This text is somewhat similar in treatment to that of the Kimball Text, and in the hands of a good teacher will produce first class results. The authors in the preparation of the book kept three things in mind, we are told: First, to present the fundamental facts of physics in a clear, concise and teachable form; second, to relate these fundamental facts to the basic laws and to the theories of physics in such way as to render plain the historical growth of the science; and, third, to put the student in direct touch with first hand information concerning the epoch-making discoveries of the past upon which the growth of the science has been based, as well as to afford an intimation of the marvelous progress of the present. F.

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NOTE ON THE DERIVATIVE OF THE QUOTIENT OF TWO WRONSKIANS.

By L. A. HOWLAND, Middletown, Connecticut.

Suppose we have given a set of $n+1$ analytic functions of x, in which there are two sets of n functions each, which are linearly independent. A formula for the derivative of the quotient of the Wronskians of these two sets has important application in the theory of linear differential equations and elsewhere.* The formula is supposed to be due to Frobenius, who derived and applied it in the article cited. It may be of interest to note that we can reverse the application and use the theory of linear differential equations to establish the formula.

Since it is merely a question of notation, we shall assume the sets to be: (1) the first n functions, and (2) the first $n-1$ and the last. We have then to evaluate

$$(1) \qquad \frac{d}{dx}\frac{W_2(a_1, a_2, \ldots, a_{n-1}, a_{n+1})}{W_1(a_1, a_2, \ldots, a_{n-1}, a_n)} = \frac{W_1 W_2' - W_1' W_2}{W_1^2}$$

where $W_1 = W_1(a_1, a_2, \ldots, a_n) = \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ a_1' & a_2' & \ldots & a_n' \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{(n-1)} & a_1^{(n-1)} & \ldots & a_n^{(n-1)} \end{vmatrix}$

and similarly for W_2.

Form the equation:

$$(2) \qquad W_1 W_2' - W_1' W_2 - W(a_1, \ldots, a_{n-1})\, W(a_1, \ldots, a_{n+1}) = 0,$$

or written out in determinant form:

* Cf., for example, Frobenius, Crelle, 77, p. 248; Schlesinger, Handbuch d. Theorie d. Lin. Diff. Gl., Bd. 1, p. 60; Curtiss, Math. Ann., Bd. 65, p. 284; also Bull. Am. Math. Soc., Vol. XVII, p. 465.